## EXAMPLE 4.8-1: Plate with Transpiration

Transpiration is one technique for insulating a surface from an adjacent flowing fluid (e.g., a turbine blade exposed to high temperature gas). Fluid is blown through small holes in the surface so that there is a specified $y$-directed velocity at the plate surface ($v_{y=0} = v_b$).

a.) Determine the ordinary differential equation that governs the growth of the momentum boundary layer for a flat plate experiencing transpiration. Use the momentum integral technique with an assumed second order velocity distribution.

The second order velocity distribution in Table 4-4 is used and the terms associated with shear at the edge of the boundary layer are neglected. The resulting velocity distribution and surface shear stress are:

$$\frac{u}{u_\infty} = \left[2\frac{y}{\delta_m} - \frac{y^2}{\delta_m^2}\right] \tag{1}$$

and

$$\tau_s = 2\mu\frac{u_\infty}{\delta_m} \tag{2}$$

The momentum integral equation, Eq. (4-470), can be simplified for this problem:

$$\frac{d}{dx}\left[\int_{y=0}^{y=\delta_m}\left(u^2 - u\,u_\infty\right)dy\right] + v_b\,u_\infty = -\frac{\tau_s}{\rho} \tag{3}$$

Substituting Eqs. (1) and (2) into Eq. (3) leads to:

$$u_\infty^2\frac{d}{dx}\left[\int_{y=0}^{y=\delta_m}\left(\left(2\frac{y}{\delta_m} - \frac{y^2}{\delta_m^2}\right)^2 - 2\frac{y}{\delta_m} + \frac{y^2}{\delta_m^2}\right)dy\right] + v_b\,u_\infty = -2\frac{\mu\,u_\infty}{\rho\,\delta_m}$$

or

$$u_\infty^2\frac{d}{dx}\left[\int_{y=0}^{y=\delta_m}\left(-2\frac{y}{\delta_m} + 5\frac{y^2}{\delta_m^2} - 4\frac{y^3}{\delta_m^3} + \frac{y^4}{\delta_m^4}\right)dy\right] + v_b\,u_\infty = -2\frac{\mu\,u_\infty}{\rho\,\delta_m}$$

Carrying out the integration leads to:

$$u_\infty^2\frac{d}{dx}\left[\left(-\frac{y^2}{\delta_m} + \frac{5}{3}\frac{y^3}{\delta_m^2} - \frac{y^4}{\delta_m^3} + \frac{1}{5}\frac{y^5}{\delta_m^4}\right)_0^{\delta_m}\right] + v_b\,u_\infty = -2\frac{\mu\,u_\infty}{\rho\,\delta_m}$$

Applying the limits leads to:

$$-\frac{2}{15}u_\infty^2\frac{d\delta_m}{dx}+v_b\,u_\infty=-2\frac{\mu\,u_\infty}{\rho\,\delta_m} \tag{4}$$

Solving for the rate of change of the momentum boundary layer:

$$\frac{d\delta_m}{dx}=\frac{15}{2\,u_\infty}\left(2\frac{\mu}{\rho\,\delta_m}+v_b\right) \tag{5}$$

which is the ordinary differential equation that governs the boundary layer growth.

This derivation can also be accomplished using Maple. The assumed velocity distribution and shear stress, Eqs. (1) and (2), are entered:

```
> restart;
> u:=u_infinity*(2*y/delta_m(x)-y^2/delta_m(x)^2);
```

$$u := u\_infinity\left(\frac{2\,y}{\mathrm{delta\_m}(x)}-\frac{y^2}{\mathrm{delta\_m}(x)^2}\right)$$

```
> tau_s:=2*mu*u_infinity/delta_m(x);
```

$$tau\_s := \frac{2\,\mu\,u\_infinity}{\mathrm{delta\_m}(x)}$$

The momentum integral equation, Eq. (3), is entered.

```
> ODE:=diff(int(u^2-u*u_infinity,y=0..delta_m(x)),x)+v_b*u_infinity=-tau_s/rho;
```

$$ODE := -\frac{2}{15}\,u\_infinity^2\left(\frac{d}{dx}\,\mathrm{delta\_m}(x)\right)+v\_b\,u\_infinity=-\frac{2\,\mu\,u\_infinity}{\mathrm{delta\_m}(x)\,\rho}$$

The result identified by Maple is identical to Eq. (4). The rate of change of the momentum boundary layer is obtained using the solve command.

```
> ddeltamdx:=solve(ODE,diff(delta_m(x),x));
```

$$ddeltamdx := \frac{15}{2}\,\frac{v\_b\,\mathrm{delta\_m}(x)\,\rho+2\,\mu}{u\_infinity\,\mathrm{delta\_m}(x)\,\rho}$$

The result identified by Maple is identical to Eq. (5)

b.) Use a numerical method to obtain a solution for the local friction coefficient as a function of Reynolds number. The plate is $L$ = 0.2 m long and the fluid has properties $\rho$ = 10 kg/m$^3$ and $\mu$ = 0.0005 Pa-s. The free stream velocity is $u_\infty$ = 10 m/s.

The inputs are entered in EES:

```
"EXAMPLE 4.8-1: Flat Plate with Transpiration"
$UnitSystem SI MASS RAD PA K J
$TABSTOPS   0.2 0.4 0.6 0.8 3.5 in

"Inputs"
L=0.2 [m]                                   "length of plate"
v_b=0.1 [m/s]                               "blowing velocity"
rho=10 [kg/m^3]                             "density of fluid"
mu=0.0005 [Pa-s]                            "viscosity of fluid"
u_infinity=10 [m/s]                         "velocity of free stream"
```

The variation of the boundary layer thickness with position is the solution to the ordinary differential equation, Eq. (5), subject to the initial condition:

$$\delta_{m,x=0} = 0 \tag{6}$$

The rate of change of the boundary layer thickness with position is infinite at $x = 0$ if Eq. (6) is substituted into Eq. (5). This characteristic can result in some difficulties if Eq. (5) is integrated numerically. Therefore, it is difficult to start the numerical integration. One approach is to specify a small but non-zero boundary layer thickness at the leading edge of the plate and integrate from that initial condition. A more sophisticated and reliable technique recognizes that the first term in Eq. (5) dominates near the leading edge of the plate. Therefore, very near the leading edge the ordinary differential equation, Eq. (5), becomes:

$$\frac{d\delta_m}{dx} \approx 15 \frac{\mu}{u_\infty \rho \delta_m}$$

which can be integrated analytically rather than numerically from $x = 0$ to $x = x_{si}$ where $x_{si}$ is a position sufficiently removed from the leading edge that both terms in Eq. (5) must be considered:

$$\int_0^{\delta_{m,si}} \delta_m \, d\delta_m = 15 \frac{\mu}{u_\infty \rho} \int_0^{x_{si}} dx$$

which leads to:

$$\delta_{m,si} = \sqrt{\frac{30 \mu x_{si}}{u_\infty \rho}} \tag{7}$$

The numerical integration will therefore start from $x = x_{si}$ and $\delta_m = \delta_{m,si}$ rather than at $x = 0$ and $\delta_m = 0$. The starting point for the numerical integration, $x_{si}$, should be selected based on the location where the second term in Eq. (5) becomes significant in relation to the first term. The ratio of the second to the first terms of Eq. (5) is:

$$\frac{2^{nd} \text{ term in Eq. (5)}}{1^{st} \text{ term in Eq. (5)}} = \frac{v_b \, \rho \, \delta_m}{2 \, \mu} \tag{8}$$

The starting point for the integration will be selected so that the ratio in Eq. (8) reaches a value of 0.01 at $x_{si}$. Substituting Eq. (7) into Eq. (8) leads to:

$$\frac{v_b \, \rho \, \delta_{m,si}}{2 \, \mu} = \frac{v_b \, \rho}{2 \, \mu} \sqrt{\frac{30 \, \mu \, x_{si}}{u_\infty \, \rho}} = 0.01 \tag{9}$$

Solving Eq. (9) for $x_{si}$ leads to:

$$x_{si} = \frac{4 \left(0.01\right)^2}{30} \frac{\mu \, u_\infty}{v_b^2 \, \rho}$$

```
x_si=4*(0.01)^2*mu*u_infinity/(30*v_b^2*rho)            "starting point for integration"
delta_m_si=sqrt(30*mu*x_si/(u_infinity*rho))            "boundary layer thickness at the starting point"
```

Equation (5) is entered in EES and the Integral command is used to integrate the boundary layer from $x = x_{si}$ to $x = L$:

```
ddelta_mdx=15*(2*mu/(rho*delta_m)+v_b)/(2*u_infinity)  "rate of change of boundary layer with position"
delta_m=delta_m_si+integral(ddelta_mdx,x,x_si,L)        "integral solution"
```

The surface shear stress ($\tau_s$) is computed according to Eq. (2):

```
tau_s=2*mu*u_infinity/delta_m   "shear stress"
```

The Reynolds number is calculated according to:

$$Re_x = \frac{\rho \, u_\infty \, x}{\mu}$$

and the friction factor is calculated according to:

$$C_f = \frac{2 \, \tau_s}{\rho \, u_\infty^2}$$

```
Re_x=rho*u_infinity*x/mu                  "Reynolds number"
C_f=2*tau_s/(rho*u_infinity^2)            "friction factor"
```

The analytical solution for the friction factor over a flat plate without transpiration was obtained from the Blasius solution ($C_{f,bs}$) in Section 4.4.2.

$$C_{f,bs} = \frac{0.664}{\sqrt{Re_x}}$$

```
C_f_bs=0.664/sqrt(Re_x)          "friction factor from Blasius solution"
```

The quantities are included in an integral table.

```
DELTAx=L/500                                            "spacing in integral table"
$integraltable x:DELTAx,delta_m,tau_s,Re_x,C_f,C_f_bs
```

The spacing in the integral table is specified by the variable DELTAx; however, DELTAx has no impact on the spatial step used in the integration.

Figure 1 illustrates $C_f$ and $C_{f,bs}$ as a function of $Re_x$ for various values of the blowing velocity. Notice that the integral solution agrees well with the Blasius solution when $v_b$ approaches zero and that the friction coefficient is reduced by transpiration because the boundary layer is increased.

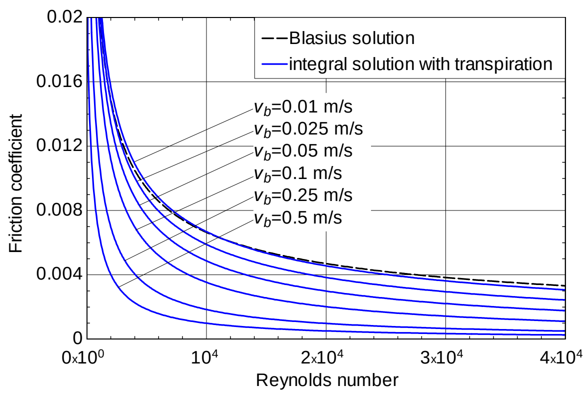


**Figure 1: Friction coefficient as a function of the Reynolds number for various values of the blowing velocity; also shown is the Blasius solution.**