Kelvin Functions[1]

The solution to the Bessel equation $x^2 w'' + xw' - ix^2 w = 0$, where $i = \sqrt{-1}$, may be written in terms of modified Bessel functions as

$$w(x) = c_1 I_0(x\sqrt{i}) + c_2 K_0(x\sqrt{i})$$

The Kelvin functions, $\text{ber}(x)$, $\text{bei}(x)$, $\text{ker}(x)$ and $\text{kei}(x)$, may be defined by their relations to the modified Bessel functions $I_0(x)$ and $K_0(x)$ as given by

$$I_0(x\sqrt{i}) = \text{ber}(x) + i\,\text{bei}(x) \qquad \text{and} \qquad K_0(x\sqrt{i}) = \text{ker}(x) + i\,\text{kei}(x)$$

Thus it follows that $w(x)$ may be written as

$$w(x) = c_1[\text{ber}(x) + i\,\text{bei}(x)] + c_2[\text{ker}(x) + i\,\text{kei}(x)]$$

where the four Kelvin functions are real functions. These Kelvin functions may be evaluated from the following infinite series:

$$\text{ber}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{2^{4n}[(2n)!]^2} \tag{1}$$

$$\text{bei}(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+2}}{2^{4n+2}[(2n+1)!]^2} \tag{2}$$

$$\text{ker}(x) = \frac{\pi}{4}\text{bei}(x) - \left[\gamma + \ln\left(\frac{x}{2}\right)\right]\text{ber}(x) + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n}}{2^{4n}[(2n)!]^2} \sum_{m=1}^{2n} \frac{1}{m} \tag{3}$$

$$\text{kei}(x) = -\frac{\pi}{4}\text{ber}(x) - \left[\gamma + \ln\left(\frac{x}{2}\right)\right]\text{bei}(x) - \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-2}}{2^{4n-2}[(2n-1)!]^2} \sum_{m=1}^{2n-1} \frac{1}{m} \tag{4}$$

where γ is Euler's constant given by Abramowitz and Stegun[2] as

$$\gamma = \lim_{m\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{m} - \ln(m)\right] = 0.5772156649015328606065 12$$

Abramowitz and Stegun tabulate the above four Kelvin functions for $0 \le x \le 5$. These Kelvin functions have the following characteristics:

$\text{ber}(x)$: ber(0) = 1 and oscillates with increasing amplitude as x increases
$\text{bei}(x)$: bei(0) = 0 and oscillates with increasing amplitude as x increases
$\text{ker}(x)$: ker(0) = ∞ and oscillates with decreasing amplitude as x increases
$\text{kei}(x)$: kei(0) = $-\pi/4$ and oscillates with decreasing amplitude as x increases

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1 This documentation of Kelvin functions and *KelvinFunctions.LIB* were provided by G. E. Myers, Department of Mechanical Engineering, University of Wisconsin-Madison, 2003.

2 Abramowitz, M. and I. A. Stegun (eds.): *Handbook of Mathematical Functions*, Applied Mathematics Series 55, National Bureau of Standards, 1964. [Dover, 1965].

The first derivatives of $\text{ber}(x)$, $\text{bei}(x)$, $\text{ker}(x)$ and $\text{kei}(x)$ are called $\text{ber}'(x)$, $\text{bei}'(x)$, $\text{ker}'(x)$ and $\text{kei}'(x)$, respectively. These functions, found by differentiating (1) through (4), are given by

$$\text{ber}'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-1}}{2^{4n-1}(2n-1)!(2n)!} \tag{5}$$

$$\text{bei}'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n+1}}{2^{4n+1}(2n)!(2n+1)!} \tag{6}$$

$$\text{ker}'(x) = \frac{\pi}{4}\text{bei}'(x) - \left[\gamma + \ln\left(\frac{x}{2}\right)\right]\text{ber}'(x) - \frac{1}{x}\text{ber}(x) + \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-1}}{2^{4n-1}(2n-1)!(2n)!} \sum_{m=1}^{2n} \frac{1}{m} \tag{7}$$

$$\text{kei}'(x) = -\frac{\pi}{4}\text{ber}'(x) - \left[\gamma + \ln\left(\frac{x}{2}\right)\right]\text{bei}'(x) - \frac{1}{x}\text{bei}(x) - \sum_{n=1}^{\infty} \frac{(-1)^n x^{4n-3}}{2^{4n-3}(2n-2)!(2n-1)!} \sum_{m=1}^{2n-1} \frac{1}{m} \tag{8}$$

Rather than tabulating these four derivatives, Abramowitz and Stegun tabulate $\text{ber}_1(x)$, $\text{bei}_1(x)$, $\text{ker}_1(x)$ and $\text{kei}_1(x)$ which are related to $\text{ber}'(x)$, $\text{bei}'(x)$, $\text{ker}'(x)$ and $\text{kei}'(x)$ as follows:

$$\sqrt{2}\text{ber}_1(x) = \text{ber}'(x) - \text{bei}'(x) \qquad \sqrt{2}\text{bei}_1(x) = \text{ber}'(x) + \text{bei}'(x)$$

$$\sqrt{2}\text{ker}_1(x) = \text{ker}'(x) - \text{kei}'(x) \qquad \sqrt{2}\text{kei}_1(x) = \text{ker}'(x) + \text{kei}'(x)$$

The *EES* user library contains *KelvinFunctions.LIB*. The suite of eight functions in *KelvinFunctions.LIB* are used to calculate values of $\text{ber}(x)$, $\text{bei}(x)$, $\text{ker}(x)$, $\text{kei}(x)$, $\text{ber}'(x)$, $\text{bei}'(x)$, $\text{ker}'(x)$ and $\text{kei}'(x)$. These values are calculated from (1) through (8), respectively.

The following *EES* program will evaluate $\text{ker}(x)$ and $\text{ker}'(x)$ for $x = 2$:

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x = 2
a = Kelvin_ker(x)
b = Kelvin_ker`(x)
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The solution is given as

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a = -0.04166   b = -0.1066    x = 2.0
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